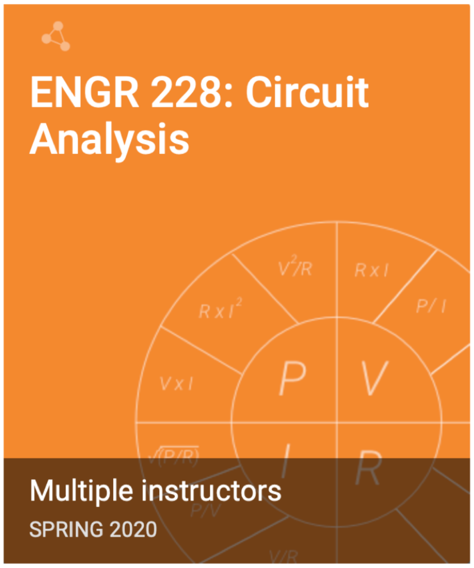


Chapter 6.2

Series RLC Circuits

Engr228 - Circuit Analysis

Spring 2020

Dr Curtis Nelson

Section 6.2 Objective

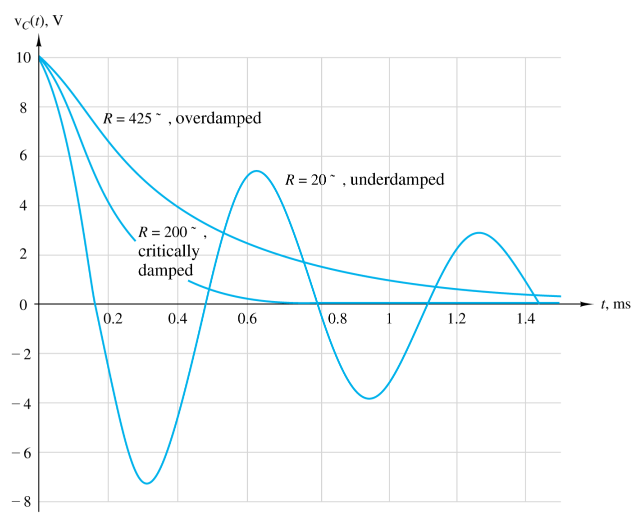
- Be able to determine the natural and step responses of series RLC circuits.

Equations for Analysing the Natural Response of Parallel *RLC* Circuits

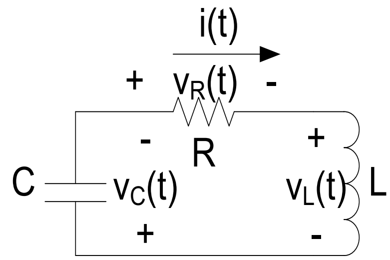
Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$
	$v(0^+) = A_1 + A_2 = V_0$
	$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t, t \geq 0$
	$v(0^+) = B_1 = V_0$
	$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, t \geq 0$
	$v(0^+) = D_2 = V_0$
	$\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Summary of Transient Responses



Source - Free Series RLC Circuit



$$v_R + v_L + v_C = 0$$

$$i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = 0$$

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

Comparing Series and Parallel RLC Circuits

Parallel RLC	Series RLC
$C\frac{d^2 v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$	$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$
$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$	$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$
$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
$\alpha = \frac{1}{2RC}$ $\quad \omega_0 = \frac{1}{\sqrt{LC}}$	$\alpha = \frac{R}{2L}$ $\quad \omega_0 = \frac{1}{\sqrt{LC}}$

Series RLC Circuit Solution

$$\omega_0 = 1/\sqrt{LC} \qquad \alpha = \frac{R}{2L} \qquad \begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}$$

If:

$\alpha > \omega_0$ (overdamped): $i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$\alpha = \omega_0$ (critically damped): $i(t) = e^{-\alpha t}\left(A_1 t + A_2\right)$

$\alpha < \omega_0$ (underdamped): $i(t) = e^{-\alpha t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right)$

Equations for Analysing the Natural Response of Series *RLC* Circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$ $i(0^+) = A_1 + A_2 = I_0$ $\frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 < \omega_0^2$: underdamped	$i(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t, \quad t \geq 0$ $i(0^+) = B_1 = I_0$ $\frac{di(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 = \omega_0^2$: critically damped	$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0$ $i(0^+) = D_2 = I_0$ $\frac{di(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{L}(-RI_0 - V_0)$

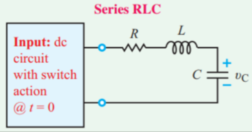
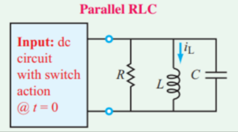
(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Equations for Analysing the Step Response of Series *RLC* Circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = \frac{V}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v_C(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}, t \geq 0$ $v_C(0^+) = V_f + A_1' + A_2' = V_0$ $\frac{dv_C(0^+)}{dt} = s_1 A_1' + s_2 A_2' = \frac{I_0}{C}$
$\alpha^2 < \omega_0^2$: underdamped	$v_C(t) = V_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, t \geq 0$ $v_C(0^+) = V_f + B_1' = V_0$ $\frac{dv_C(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{I_0}{C}$
$\alpha^2 = \omega_0^2$: critically damped	$v_C(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, t \geq 0$ $v_C(0^+) = V_f + D_2' = V_0$ $\frac{dv_C(0^+)}{dt} = D_1' - \alpha D_2' = \frac{I_0}{C}$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Zybooks Response Summary

Series RLC	Parallel RLC
	
Total Response	**Total Response**
Overdamped ($\alpha > \omega_0$) $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty)$ $A_1 = \frac{\frac{1}{C} i_C(0) - s_2[v_C(0) - v_C(\infty)]}{s_1 - s_2}$ $A_2 = \left[\frac{\frac{1}{C} i_C(0) - s_1[v_C(0) - v_C(\infty)]}{s_2 - s_1}\right]$	**Overdamped** ($\alpha > \omega_0$) $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + i_L(\infty)$ $A_1 = \frac{\frac{1}{L} v_L(0) - s_2[i_L(0) - i_L(\infty)]}{s_1 - s_2}$ $A_2 = \left[\frac{\frac{1}{L} v_L(0) - s_1[i_L(0) - i_L(\infty)]}{s_2 - s_1}\right]$
Critically Damped ($\alpha = \omega_0$) $v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$ $B_1 = v_C(0) - v_C(\infty)$ $B_2 = \frac{1}{C} i_C(0) + \alpha[v_C(0) - v_C(\infty)]$	**Critically Damped** ($\alpha = \omega_0$) $i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$ $B_1 = i_L(0) - i_L(\infty)$ $B_2 = \frac{1}{L} v_L(0) + \alpha[i_L(0) - i_L(\infty)]$
Underdamped ($\alpha < \omega_0$) $v_C(t) = e^{-\alpha t}(D_1 \cos \omega_d t + D_2 \sin \omega_d t) + v_C(\infty)$ $D_1 = v_C(0) - v_C(\infty)$ $D_2 = \frac{\frac{1}{C} i_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d}$	**Underdamped** ($\alpha < \omega_0$) $i_L(t) = e^{-\alpha t}(D_1 \cos \omega_d t + D_2 \sin \omega_d t) + i_L(\infty)$ $D_1 = i_L(0) - i_L(\infty)$ $D_2 = \frac{\frac{1}{L} v_L(0) + \alpha[i_L(0) - i_L(\infty)]}{\omega_d}$

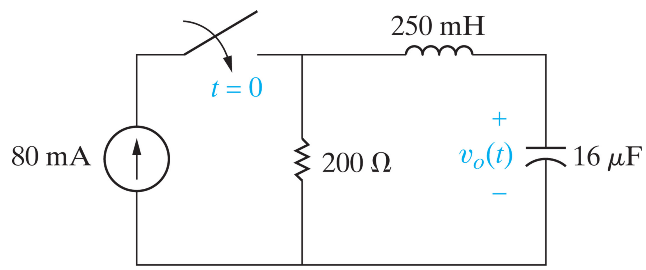
Auxiliary Relations

$$\alpha = \begin{cases} \frac{R}{2L} & \text{Series RLC} \\ \frac{1}{2RC} & \text{Parallel RLC} \end{cases} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

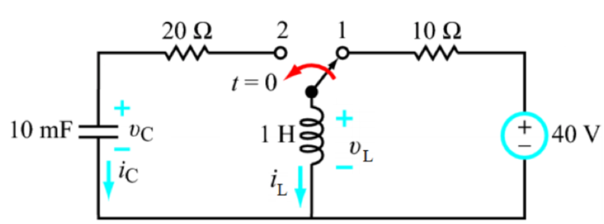
Textbook Problem 8.50 (Nilsson 11th)

The circuit contains no initial energy. Find $v_o(t)$ for $t \geq 0$.



$v_o(t) = 16 - 16e^{-400t}\cos 300t - 21.33e^{-400t}\sin 300t\ V$

Zybooks Participation Exercise 6.5.3



Summary: Solving RLC Circuits

1. Identify the series or parallel RLC circuit;
2. Find α and ω_0;
3. Determine whether the circuit is overdamped, critically damped, or underdamped;
4. Assume a solution (natural response + forced response):

$A_1e^{s_1t} + A_2e^{s_2t} + V_f$	Overdamped
$A_1te^{st} + A_2e^{st} + V_f$	Critically damped
$e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t) + V_f$	Underdamped

5. Find *A*, *B*, and V_f using initial and final conditions.

Equations for Analysing the Natural Response of Parallel *RLC* Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1e^{s_1t} + A_2e^{s_2t}, t \geq 0$
	$v(0^+) = A_1 + A_2 = V_0$
	$\frac{dv(0^+)}{dt} = s_1A_1 + s_2A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t, t \geq 0$
	$v(0^+) = B_1 = V_0$
	$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t}, t \geq 0$
	$v(0^+) = D_2 = V_0$
	$\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Equations for Analysing the Step Response of Parallel RLC Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = \frac{I}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i_L(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \quad t \geq 0$
	$i_L(0^+) = I_f + A_1' + A_2' = I_0$
	$\frac{di_L(0^+)}{dt} = s_1 A_1' + s_2 A_2' = \frac{V_0}{L}$
$\alpha^2 < \omega_0^2$: underdamped	$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, \quad t \geq 0$
	$i_L(0^+) = I_f + B_1' = I_0$
	$\frac{di_L(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{V_0}{L}$
$\alpha^2 = \omega_0^2$: critically damped	$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \quad t \geq 0$
	$i_L(0^+) = I_f + D_2' = I_0$
	$\frac{di_L(0^+)}{dt} = D_1' - \alpha D_2' = \frac{V_0}{L}$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Section 6.2 Summary

- Showed how to determine the natural and step responses of series RLC circuits.